

Superconformal Quantum Mechanics and the Discrete Light-Cone Quantisation of $\mathcal{N}=4$ SUSY Yang-Mills

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Abstract

We study the quantum mechanical σ -model arising in the discrete light-cone quantisation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The target space is a certain torus fibration over a scale-invariant special Kähler manifold. We show that the expected $SU(1, 1|4)$ light-cone superconformal invariance of the $\mathcal{N} = 4$ theory emerges in a limit where the volume of the fibre goes to zero and give an explicit construction of the generators. The construction given here yields a large new family of superconformal quantum mechanical models with $SU(1, 1|4)$ invariance.

1 Introduction

Supersymmetric gauge theories in four dimensions exhibit a wide range of interesting phenomena including a remarkable web of dualities relating seemingly different models. Many of these properties can be understood in terms of a mysterious conformal theory with $(2, 0)$ supersymmetry in six dimensions which gives rise to various 4d gauge theories after compactification [1]. The simplest case is the compactification of this $(2, 0)$ theory on a two-torus which gives $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions at low energy. Modular transformations of the torus then correspond to electric-magnetic duality transformations in the low-energy theory.

Although we know very little about the $(2, 0)$ theory, a concrete proposal [2] exists to define the theory compactified on a null circle in terms of quantum mechanics on the moduli space of Yang-Mills instantons. This is an example of the more general phenomenon of discrete light-cone quantisation (DLCQ), where restriction to a sector of fixed null momentum yields a finite dimensional quantum mechanical model. The relation of the six-dimensional theory to $\mathcal{N} = 4$ super-Yang-Mills in four dimensions also gives rise to a related proposal for the DLCQ description of the latter theory [3, 4]. The main goal of this paper is to formulate the quantum mechanical model describing the DLCQ of the $\mathcal{N} = 4$ theory explicitly and construct its symmetry algebra.

In discrete light-cone quantisation, the quantum mechanical model describing a sector of fixed null momentum inherits a subgroup of the spacetime symmetry of the full theory. In the case of a superconformal field theory, the reduced model is invariant under the subgroup of the superconformal symmetry which commutes with the null momentum. As we review in Section 2 below, this corresponds to an $SU(1, 1|4)$ subgroup of the $PSU(2, 2|4)$ superconformal invariance of the $\mathcal{N} = 4$ theory. Thus we seek an $SU(1, 1|4)$ -invariant superconformal quantum mechanics. The model proposed in [3, 4] takes the form of a quantum mechanical σ -model. As we review in Section 3 below, the target space corresponds to the moduli space of instantons in an auxiliary Yang-Mills theory living on $\mathbb{R}^2 \times T^2$. More precisely we should

consider a limit where the area of the the torus goes to zero. We will show that the expected $SU(1, 1|4)$ invariance indeed emerges in this limit.

A beautiful feature of non-linear σ -models is that the conditions for unbroken supersymmetry have a geometric character. A famous example is that $\mathcal{N} = (4, 4)$ supersymmetry in one or two dimensions requires a hyper Kähler target [5]. Similarly, superconformal invariance of a quantum mechanical σ -model also constrains the geometry of the target space [6]. Although several families of superconformal σ -models corresponding to different target space geometries are known, the conditions for $SU(1, 1|4)$ invariance have not been discussed before in the literature. Our main result is that scale-invariant target spaces with *special Kähler geometry* naturally solve these constraints. As we discuss below, the dimensional reduction of $\mathcal{N} = 2$ superconformal models in four dimensions provides a large class of interesting examples. In particular, the relevant target space for the DLCQ description of the $\mathcal{N} = 4$ theory takes the form of a torus fibration over a special Kähler manifold. In the limit relevant for $\mathcal{N} = 4$ super-Yang-Mills, the volume of the fibre goes to zero and the model is one of this class.

The paper is organised as follows. After a brief review of discrete light-cone quantisation and its application to the $\mathcal{N} = 4$ theory, we proceed to construct a quantum mechanical σ -model whose target space is a certain torus bundle over a generic scale-invariant special Kähler base. Although our main interest is the limit in which the dynamics in the fibre directions decouples giving superconformal quantum mechanics on the base, the more general model is interesting for two reasons. First, it arises in the context of DLCQ where it corresponds to the compactification of the $(2, 0)$ theory on a torus of finite area. Second, it provides a natural setting for the resolution of the singularities of the base manifold which is likely needed to make sense of these models. The bulk of the paper is devoted to studying the symmetry algebra of the full σ -model and its enhancement to $SU(1, 1|4)$ in the relevant limit. A brief discussion of singularities and their resolution is given in the final section. In this paper, we focus mainly on the general class of models described above. Discussion of the particular case relevant to the DLCQ of the $\mathcal{N} = 4$ theory will be given in a separate paper [7].

2 $\mathcal{N}=4$ SUSY Yang-Mills on the Light-Cone

We will consider $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ and complexified coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}.$$

The theory defined on $\mathbb{R}^{3,1}$ has superconformal invariance which is unbroken at the origin of the moduli space where the vacuum expectation values of all scalar fields vanish. The full superconformal group is isomorphic to $PSU(2, 2|4)$. In addition to the usual Poincaré generators P_m , M_{mn} , the bosonic part of the corresponding algebra includes the dilatation operator D and special conformal transformations K_m as well as the generators $R^{A\bar{B}}$ of the $SU(4)$ R-symmetry. The algebra is completed by Poincaré supersymmetry generators Q_α^A , $\bar{Q}_{\dot{\alpha}}^{\bar{A}}$ in the $(\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$ of $SU(2)_L \times SU(2)_R \times SU(4)$ and superconformal generators S_α^A , $\bar{S}_{\dot{\alpha}}^{\bar{A}}$ also transforming as $(\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$.

In discrete light-cone quantisation (DLCQ) one considers the theory compactified on a light-like circle. Starting from Minkowski space with Cartesian coordinates $\{x_0, x_1, x_2, x_3\}$, we define light-cone coordinates $x_+ = x_0 + x_1$ and $x_- = x_0 - x_1$ and impose the periodic identification $x_- \sim x_- + 2\pi R_-$. Thus we replace Minkowski space with the spacetime manifold

$$\mathcal{M}_4 = \mathbb{R}^2 \times S_-^1 \times \mathbb{R}_+,$$

where x_\pm are coordinates on \mathbb{R}_+ and S_-^1 respectively. The momentum $p_+ = p_0 + p_1$ conjugate to x_- , which is manifestly positive for on-shell states, is thus quantised as $p_+ = K/R_-$ where K is a positive integer. In light-cone quantisation the coordinate x_+ plays the role of time and the conjugate momentum $p_- = p_0 - p_1$ is the corresponding Hamiltonian. Working in a sector of fixed K typically reduces the field theory to a finite-dimensional quantum mechanics model.

Although light-like compactification on a circle of fixed radius breaks dilatation symmetry of the four-dimensional theory, a linear combination of scaling with a Lorentz boost in the compact direction remain unbroken. The corresponding generator $T = D + M_{01}$ is

known as the light-cone dimension. More generally, compactification on \mathcal{M}_4 breaks the four-dimensional superconformal group down to the “collinear subgroup”¹. The four-dimensional conformal group is broken as,

$$SO(4, 2) \rightarrow SO(2, 1) \times SO(2)$$

where the unbroken $SO(2, 1)$ factor is generated by the light-cone dimension $T = D + M_{01}$, the light-cone Hamiltonian $H = p_-$ and the special conformal transformation $K = K_0 + K_1$;

$$[T, K] = 2iK \quad [T, H] = -2iH \quad [H, K] = -4iT.$$

The unbroken $SO(2)$, with generator $J = M_{23}$, corresponds to rotations in the transverse \mathbb{R}^2 . In fact the DLCQ theory has a larger spacetime symmetry group known as the Schrödinger group which also contains Galilean boosts, but these extra generators do not affect the following and we will not discuss them.

Light-like compactification also breaks half of the fermionic symmetries of the four-dimensional theory. In particular each of the two-component spinor supercharges Q_α^A , $\bar{Q}_{\dot{\alpha}}^{\bar{A}}$, S_α^A and $\bar{S}_{\dot{\alpha}}^{\bar{A}}$ has a projection onto the light-cone corresponding to an unbroken symmetry (see [8] for further details). We denote the unbroken generators as Q^A , $\bar{Q}^{\bar{A}}$, S^A and $\bar{S}^{\bar{A}}$. The $SU(4)$ R-symmetry also remains unbroken and the full unbroken bosonic symmetry is therefore

$$G_B = SO(2, 1) \times SO(2) \times SU(4) \simeq SU(1, 1) \times U(4).$$

The fermionic generators Q^A , S^A form a doublet of $SU(1, 1)$ as do $\bar{Q}^{\bar{A}}$, $\bar{S}^{\bar{A}}$.

The full symmetry of DLCQ should correspond to a Lie superalgebra whose maximal bosonic subalgebra is $\text{Lie}(G_B)$ and which also includes sixteen fermionic generators in the $(\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}, \bar{\mathbf{4}})$ of G_B . Up to automorphisms, the unique possibility is $SU(1, 1|4)$. The DLCQ of $\mathcal{N} = 4$ SUSY Yang-Mills should therefore be superconformal quantum mechanics with this symmetry. The main goal of this paper is to identify this model and construct its symmetry generators explicitly.

¹More precisely the collinear subgroup described eg in [8] contains an extra generator which preserves the light-cone but not the radius of the light-like compactification.

The light-like compactification described above also allows one to introduce Wilson lines for the $SU(N)$ gauge field

$$\left\langle \oint_{S^1_-} A \cdot dx \right\rangle = \text{diag} \{ \mu_1, \mu_2, \dots, \mu_N \}, \quad \sum_{i=1}^N \mu_i = 0.$$

If $\mu_i \neq \mu_j$ for all i and j then the gauge group is broken down to its Cartan subalgebra by the adjoint Higgs mechanism:

$$SU(N) \rightarrow U(1)^{N-1}.$$

Further, performing a duality transformation on the resulting three-dimensional abelian low-energy effective theory, one can also introduce corresponding magnetic Wilson lines denoted ρ_i for $i = 1, 2, \dots, N$. The electric and magnetic Wilson lines are naturally combined to form N complex parameters, $Z_i = \rho_i + \tau \mu_i$. Taking into account the standard 2π -periodicity of the Wilson lines, the $\{Z_i\}$ correspond to N points on a torus of complex structure τ . For gauge group $SU(N)$, only the $N - 1$ relative positions of these points are significant.

Importantly neither the electric or magnetic Wilson lines break the light-cone superconformal symmetry of DLCQ identified above. Thus we seek a quantum mechanical model with $SU(1, 1|4)$ superconformal symmetry and $N - 1$ additional complex parameters.

3 DLCQ of $\mathcal{N}=4$ SUSY Yang-Mills from Six Dimensions

The approach to the $\mathcal{N} = 4$ theory which we take here starts by realising the theory as a compactification of six-dimensional conformal field theory. In particular $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ arises as a low-energy effective theory when the $(2, 0)$ superconformal field theory of type A_{N-1} is compactified down to four dimensions on a two-dimensional torus [1]. The complex structure parameter of the torus coincides with the complexified coupling $\tau = 4\pi i/g^2 + \theta/2\pi$ of the $\mathcal{N} = 4$ theory. If the torus has area \mathcal{A} the full theory also contains an infinite tower of Kaluza-Klein (KK) modes corresponding to states carrying momentum along the two compact dimensions. In the limit $\mathcal{A} \rightarrow 0$ the KK modes decouple and the remaining theory is precisely $\mathcal{N} = 4$ super-Yang-Mills.

The $(2, 0)$ theory in six non-compact dimensions has a well established DLCQ description [2, 9]. Following our discussion of the four-dimensional theory above, we compactify the $(2, 0)$ theory on

$$\mathcal{M}_6 = \mathbb{R}^4 \times S_-^1 \times \mathbb{R}_+,$$

where S_-^1 is a light-like circle of radius R_- . The sector of the theory with K units of momentum in the compact direction is described by supersymmetric quantum mechanics on the moduli space of K Yang-Mills instantons of gauge group $SU(N)$ on \mathbb{R}^4 . The instanton moduli space is a hyper-Kähler manifold of real dimension $4KN$. A quantum mechanical σ -model with a hyper-Kähler target admits an $\mathcal{N} = (4, 4)$ supersymmetric extension. In fact the instanton moduli space is also equipped with a triholomorphic homothety of degree two. Under these conditions $\mathcal{N} = (4, 4)$ supersymmetry is enlarged to give an $OSp(4|4)$ superconformal invariance [10]. The latter coincides with the subgroup of the $(2, 0)$ superconformal algebra in six dimensions left unbroken by compactification on \mathcal{M}_6 .

To obtain a DLCQ description of the $\mathcal{N} = 4$ theory it is necessary to compactify two of the transverse dimensions on a torus. Thus we consider the $(2, 0)$ theory compactified on

$$\tilde{\mathcal{M}}_6 = \mathbb{R}^2 \times T_\tau^2 \times S_-^1 \times \mathbb{R}_+.$$

As above the complex structure parameter of the torus, denoted τ , is identified with the complexified coupling of the four-dimensional gauge theory. The resulting description of the sector with K units of momentum along S_-^1 is again a quantum mechanical σ -model with $\mathcal{N} = (4, 4)$ supersymmetry. The model, which was introduced in [3, 4], has as its target space the moduli space of K instantons in $SU(N)$ Yang-Mills theory living on $\mathbb{R}^2 \times T_\tau^2$. In the following we will denote this manifold as $\mathcal{M}_{K,N}$. To obtain the DLCQ of the $\mathcal{N} = 4$ theory we should take the area \mathcal{A} of the torus to zero holding its shape fixed.

The moduli space $\mathcal{M}_{K,N}$ is again a hyper-Kähler manifold of real dimension² $4KN$. Although, for general values of the parameters \mathcal{A} and τ , the hyper-Kähler metric is not known explicitly, the manifold $\mathcal{M}_{K,N}$ has (at least) two useful descriptions. The first description arises via the ADHM Nahm transform which maps $\mathcal{M}_{K,N}$ to the moduli space of Hitchin's

²More precisely, as we discuss below, the metric becomes singular along $4N - 4$ real directions.

equations on \hat{T}_τ^2 in the presence of punctures at the points $z = Z_i$, for $i = 1, 2, \dots, N$, corresponding to the electric and magnetic Wilson lines discussed in the previous section. In more physical language, the moduli space can be thought of as the Higgs branch of an auxiliary supersymmetric gauge theory on \hat{T}_τ^2 with localised impurities at the punctures [4].

The second description of $\mathcal{M}_{K,N}$ arises via the three-dimensional mirror symmetry [11] which maps the Higgs branch of the impurity theory to the Coulomb branch of yet another auxiliary supersymmetric gauge theory [12]. The starting point for the Coulomb branch description is a four-dimensional quiver gauge theory with $\mathcal{N} = 2$ supersymmetry. The quiver diagram for the theory in question coincides with the Dynkin diagram for the affine Lie algebra \hat{A}_{N-1} . The gauge group is

$$\hat{G} = U(K)_1 \times U(K)_2 \times \dots \times U(K)_N.$$

In addition to an $\mathcal{N} = 2$ vector multiplet for each $SU(K)$ factor in \hat{G} , the theory contains hypermultiplets in the bifundamental representation of adjacent factors. Thus we have a hypermultiplet in the $(\bar{\mathbf{k}}, \mathbf{k})$ of $U(K)_i \times U(K)_{i+1}$ for $i = 1, 2, \dots, N$ with the identification $U(K)_{N+1} \simeq U(K)_1$. An important subtlety is that the gauge coupling of the $U(1)$ center of each of the $N - 1$ off-diagonal $U(K)$ factors in \hat{G} has a positive β function and therefore exhibits a Landau pole. The effect is to freeze out each of these $U(1)$ factors to give a theory with gauge group

$$\hat{G}' = U(1)_D \times \prod_{j=1}^N SU(K)_j, \quad (3.1)$$

where $U(1)_D$ corresponds to the center of the diagonal $U(K)$ in the original gauge group \hat{G} . The β functions for the remaining gauge couplings vanish and the resulting theory is an $\mathcal{N} = 2$ superconformal field theory in four dimensions. The complexified coupling for the diagonal $U(K)$ is identified with the coupling, τ , of the original $\mathcal{N} = 4$ theory. The gauge couplings for the remaining off-diagonal $SU(K)$ factors encode the electric and magnetic Wilson lines of the DLCQ theory as $\tau_i = (Z_{i+1} - Z_i)/2\pi i$ for $i = 1, 2, \dots, N$ with the identification $Z_{N+1} = Z_1$.

The resulting $\mathcal{N} = 2$ theory is precisely the elliptic quiver theory first solved in [13]. The theory has a Coulomb branch of complex dimension $KN - N + 1$ parametrized by the VEVs

of the complex scalars in the vector multiplet of \hat{G}' . The metric on the Coulomb branch is determined by the corresponding Seiberg-Witten curve Σ and meromorphic differential λ for which we will not need explicit forms in the following. In fact, the considerations of this paper apply to the Coulomb branch of any $\mathcal{N} = 2$ superconformal theory in four dimensions and we will review the general features of these models in the next section. In particular, the four-dimensional Coulomb branch is a special Kähler manifold corresponding to the complex structure moduli space $\mathcal{M}(\Sigma)$ of the Seiberg-Witten curve. As the theory is conformal, the Coulomb branch metric is also scale-invariant.

In order to obtain our target space $\mathcal{M}_{K,N}$ we are instructed [12] to compactify the four-dimensional $\mathcal{N} = 2$ theory described above down to three dimensions on a circle of radius $R \sim 1/\mathcal{A}$. As we review in the next section, the Coulomb branch of the compactified theory acquires additional dimensions corresponding to electric and magnetic Wilson lines on the circle. The resulting space is a fibration of the Jacobian torus $\mathcal{J}(\Sigma)$ over the original Coulomb branch $\mathcal{M}(\Sigma)$ of the four-dimensional theory. The scale R enters as the inverse volume of the fibre. The total space of the fibration is a hyper-Kähler manifold of real dimension $4(KN - N + 1)$ which is identified with $\mathcal{M}_{K,N}$. Importantly, although the metric on $\mathcal{M}_{K,N}$ is hard to describe in general, it approaches a simple analytic form known as the *semi-flat metric* in a suitable limit where R goes to infinity [14, 15].

In the following we will consider a quantum mechanical σ -model with target space $\mathcal{M}_{K,N}$ using the “Coulomb branch” description of this manifold reviewed above. As the target space is hyper-Kähler the corresponding σ -model has $\mathcal{N} = (4, 4)$ supersymmetry. However, the compact fibre has a fixed volume set by the scale $R \sim 1/\mathcal{A}$. Thus the resulting σ -model cannot be conformally-invariant. This is consistent with its interpretation as the DLCQ of the $(2, 0)$ theory on a torus of fixed area \mathcal{A} . However, if we are to obtain a DLCQ description of the $\mathcal{N} = 4$ theory in the limit $\mathcal{A} \rightarrow 0$, then it must be that $SU(1, 1|4)$ superconformal invariance emerges in this limit. In fact, the model in question is part of a large family which can be obtained by compactifying $\mathcal{N} = 2$ superconformal field theories down to three dimensions. Each of these models give rise to a Coulomb branch which takes the form of a torus fibration over a scale-invariant special Kähler manifold. We will show that $SU(1, 1|4)$ invariance emerges as required in any model of this type.

4 Special Kähler Geometry in Supersymmetric Gauge Theory

From now on we will consider the general class of models to which the quantum mechanical σ -model described above belongs. We start by considering a generic $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions with gauge group of rank r , and give a brief review of how the local³ description of special Kähler geometry emerges in this context, as well as in the compactification of these theories on $\mathbb{R}^3 \times S^1$.

The potential for the scalars in the vector multiplet has flat directions admitting a moduli space \mathcal{M} of vacua known as the Coulomb branch where the gauge group is broken down to its Cartan subgroup $U(1)^r$ by the Higgs mechanism. The low-energy theory thus includes r massless photons⁴ A_m^I , with field strength v_{mn}^I for $I = 1, 2, \dots, r$, and their $\mathcal{N} = 2$ superpartners. In particular there are r massless complex scalar fields a^I whose vacuum expectation values provide coordinates on \mathcal{M} .

The general form of the low-energy effective action on the Coulomb branch is already highly constrained by supersymmetry [17, 18]. The bosonic part of the action must be of the form

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \tau_{IJ} \partial_m a^I \partial^m \bar{a}^J + \frac{1}{8\pi} \text{Im} \tau_{IJ} v_{mn}^I v^{Jmn} + \frac{1}{8\pi} \text{Re} \tau_{IJ} v_{mn}^I \tilde{v}^{Jmn}. \quad (4.1)$$

Here $\tau_{IJ}(a)$ is the matrix of complexified gauge couplings, and $\mathcal{N} = 2$ supersymmetry forces

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \quad (4.2)$$

with $\mathcal{F}(a)$ a holomorphic function known as the prepotential.

As usual, the coefficient matrix of the scalar kinetic terms in (4.1) defines a natural metric on \mathcal{M} :

$$ds^2 = \text{Im} \tau_{IJ} da^I d\bar{a}^J.$$

³There is a related but different notion of “local special Kähler geometry” used in supergravity [16]. In our case, “local” simply means that the description makes sense only on some coordinate patch.

⁴Here m and n denote four-dimensional Lorentz indices.

As argued in [19], this cannot be a good global description because the harmonic function $\text{Im } \tau$ is unbounded below, leading to an indefinite metric. In fact, Seiberg and Witten [19, 20] were able to give a global description allowing them to compute the exact quantum prepotential. The construction relies on an identification of the Coulomb branch with the moduli space $\mathcal{M}(\Sigma)$ of a certain family of complex algebraic curves Σ of genus r , whose period matrices correspond to the couplings τ_{IJ} . A recent pedagogical review of these curves is given in [21].

The structure summarised above, namely a complex manifold \mathcal{M} with a special holomorphic coordinate system a^I such that the metric can be expressed in terms of a prepotential \mathcal{F} as in (4.2), is known as special Kähler geometry. Note in particular that such a space is indeed always Kähler, with potential

$$K = \text{Im} \left(\frac{\partial \mathcal{F}}{\partial a^I} \bar{a}^I \right).$$

Now consider compactifying the above class of theories on $\mathbb{R}^3 \times S_R^1$ as in [14]. Here R is the radius of S^1 and we will be particularly interested in the case when R is much larger than any other length scales in the problem. In the context of the DLCQ model of Section 3, this corresponds to the limit where the size of the torus on which the $(2, 0)$ theory is defined goes to zero and we are left with $\mathcal{N} = 4$ supersymmetric Yang-Mills. Fortunately it is in this limit where the structure of the Coulomb branch is simplest and can be understood by compactifying the four-dimensional low-energy theory (4.1). In addition to the complex scalars a^I the compactified theory also contains new real periodic scalars $(\theta_e^I, \theta_{m,I})$ corresponding to the $U(1)^r$ electric and magnetic Wilson lines around S^1 . These parameterise a complex torus T^{2r} which can be identified with the Jacobian

$$\mathcal{J}(\Sigma) = \frac{\mathbb{C}^r}{\mathbb{Z}^r \oplus \tau \mathbb{Z}^r}$$

of the Seiberg-Witten curve Σ . Following [15] we define a complex coordinate

$$z_I = \theta_{m,I} - \tau_{IJ} \theta_e^J$$

and 1-form⁵

$$\delta z_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J,$$

⁵While this form is closed and equal to dz_I on $\mathcal{J}(\Sigma)$, this is no longer true on the full Coulomb branch. We will have more to say about this in section 6.

in terms of which the metric on $\mathcal{J}(\Sigma)$ is

$$ds^2 = \frac{1}{4\pi^2 R} (\text{Im } \tau^{-1})^{IJ} \delta z_I \delta \bar{z}_J.$$

The full Coulomb branch is therefore the total space of a fibre bundle⁶ $\mathcal{B} \rightarrow \mathcal{M}$ over the special Kähler base \mathcal{M} whose fibres are the Jacobian tori \mathcal{J} . When R is much larger than any other scales in the problem the metric on the total space takes its *semi-flat* form:

$$G = R \text{Im } \tau_{IJ} da^I d\bar{a}^J + \frac{1}{4\pi^2 R} (\text{Im } \tau^{-1})^{IJ} \delta z_I \delta \bar{z}_J. \quad (4.3)$$

As we review below, this is a hyper-Kähler metric as required by supersymmetry. Away from the regime of large R , the metric is considerably more complicated. In particular, the metric, which remains hyper-Kähler for any R , receives instanton corrections of order $\exp(-M_{\text{BPS}} R)$ where M_{BPS} are the masses of the BPS states of the four-dimensional theory⁷. These corrections play an important role in resolving the singularities of the semi-flat metric. There is a twistorial approach which yields integral equations determining the exact metric [15] but we will not need it here.

5 Superconformal Quantum Mechanics

The possible symmetries of quantum mechanical models with generic curved target spaces are strongly constrained by the presence of additional geometric structures in the target [5, 6, 10, 22]. To make this paper self-contained, we will now give a brief review of the key points.

We first consider some general features of quantum mechanical σ -models with (at least) $\mathcal{N} = (1, 1)$ supersymmetry. Such models contain fermions which satisfy canonical anticommutators of the form

$$\{\psi^\mu, \psi^{\dagger\nu}\} = g^{\mu\nu},$$

where $g_{\mu\nu}$ is the target space metric. These operators may be used to build up a Fock space in the usual fashion. By virtue of Fermi-Dirac statistics, this Fock space may be identified with

⁶This statement is not quite precise due to an additional subtlety in the global definition of fibre coordinates known as the quadratic refinement [15]. However, this will not play a role in the following.

⁷At weak coupling, the lightest charged BPS states are the W-bosons which yield corrections of order $\exp(-|a|R)$ where a is an integer linear combination of the scalar VEVs $\langle a^I \rangle$.

the exterior algebra of differential forms on the target [23, 24]. States of fermion number zero are described by ordinary functions (or zero-forms) on the target space. There is generically a pair of supercharges Q, Q^\dagger which may be represented via the exterior derivative and its adjoint, and the Hamiltonian in this context is naturally the Laplacian acting on forms:

$$\Delta = dd^\dagger + d^\dagger d.$$

Additional supersymmetries require extra structure on the target space [5]. $\mathcal{N} = (2, 2)$ is obtained if and only if the metric is Kähler, and the new supercharges are realised by splitting d into the Dolbeault operators

$$d = \partial + \bar{\partial}. \quad (5.1)$$

The fact that these objects produce the correct supersymmetry algebra may be taken as a definition of Kähler geometry [22]. Furthermore the expected $SU(2) \times U(1)$ R-symmetry emerges naturally from the Kähler identities and the associated Lefschetz action. This discussion extends naturally to $\mathcal{N} = (4, 4)$ supersymmetry via hyper-Kähler geometry [5, 22, 25], where there is a triplet of complex structures I^a and correspondingly three different decompositions of the exterior derivative as in (5.1). The corresponding R-symmetry is an $SO(5)$ action generalising the Lefschetz action and constructed by Verbitsky in [25].

The extension to superconformal invariance also fits into the geometric framework [6, 10]. Dilatations are generated by the flow of a vector field D on the target space, hence the dilatation operator acts as a Lie derivative on the Hilbert space of differential forms. In order to satisfy the rule $[D, H] = 2iH$, the coderivative $d^\dagger = (-1)^{np+n+1} * d *$ must be charged under this flow. Since d commutes with Lie derivatives, the solution is that the volume form must expand along the flow, and hence the vector D must be a homothety, satisfying

$$\mathcal{L}_D g = 2g.$$

The special conformal generator obeys the rules $[D, K] = -2iK$, $[H, K] = -iD$, for which it suffices that K is a function on the target space obeying

$$\mathcal{L}_D K = 2K, \quad D_\mu = \partial_\mu K. \quad (5.2)$$

A homothety obeying these extra constraints is called closed [6].

Adding in $\mathcal{N} = (1, 1)$ supersymmetry is straightforward. The supercharges are as above, and the superconformal charges are defined via $[K, Q] = iS$, leading to the expressions [10]

$$S = idK\wedge, \quad S^\dagger = -ii_D.$$

The closure of the $\{Q, S^\dagger\}$ relations onto the dilatation is then guaranteed by Cartan's formula for the Lie derivative. The resulting model then has $SU(1, 1|1)$ superconformal invariance.

To get extended supersymmetry it is necessary that the homothety interacts nicely with the complex structure, ensuring that the Dolbeault supercharges ∂ and $\bar{\partial}$ carry the correct dimensions. It suffices to demand that the homothety is a holomorphic vector field, $\mathcal{L}_D I = 0$. The hyper-Kähler case is similar and requires that D be triholomorphic. In order that $\{Q, [\bar{Q}, K]\}$ closes, it is also necessary that K is a Kähler potential [10]. The above bracket then produces a Kähler form, which as already discussed is a generator for the $SU(2)$ R-symmetry. In the hyper-Kähler case the existence of such a potential compatible with all three complex structures is a non-trivial requirement, but is always met in these models [10] (at least assuming the extra constraints (5.2)) thanks to a result of [26]. The resulting models on Kähler and hyper-Kähler manifolds have $U(1, 1|2)$ and (a real form of) $OSp(4|4)$ superconformal invariance respectively.

We will use these results to motivate the superconformal algebra we introduce in section 9, though the final structure will not be manifestly geometric as we will make a truncation to zero fibre momentum. It would be interesting to have a formulation in which the geometry is again made plain.

6 Hyper-Kähler Structure of the Coulomb Branch via

$$T^*\mathcal{M}$$

Our task is to understand quantum mechanics on the bundle \mathcal{B} with the semi-flat metric (4.3), for which it will be helpful to understand the special Kähler structure in a little more detail. We use Freed's definition [27] of special Kähler geometry. The defining feature is the existence of an extra real torsion-free connection ∇ on $T\mathcal{M}$ which is:

- Flat, $\nabla^2 = 0$
- Symplectic, $\nabla\omega = 0$
- “Special”, $d_\nabla I = 0$.

Here ω is the Kähler form and I the complex structure. It’s important that the final condition is not the same as $\nabla I = 0$. Indeed, if it were then ∇ would be Levi-Civita and the manifold would be locally isometric to \mathbb{C}^n [27]. Rather, the special Kähler condition may be written in components as⁸

$$\partial_{[\rho} I^\mu_{\nu]} + \Theta^\mu_{\sigma[\rho} I^\sigma_{\nu]} = 0.$$

This connection may be used to establish the existence of local holomorphic coordinates a^I and prepotential $\mathcal{F}(a)$ satisfying the characterisation of special Kähler geometry from section 4. Furthermore, being a flat connection, the only nontrivial consequences of ∇ are monodromies which turn out to reproduce those of the Coulomb branch from [19, 20]. Conversely, a choice of prepotential \mathcal{F} and corresponding special coordinates a^I is enough to specify a special Kähler structure locally [27], as we can calculate

$$\nabla \frac{\partial}{\partial a^I} = -\frac{i}{2} \frac{\partial^3 \mathcal{F}}{\partial a^I \partial a^J \partial a^K} da^J \otimes (\text{Im } \tau^{-1})^{KL} \left(\frac{\partial}{\partial a^L} - \frac{\partial}{\partial \bar{a}^L} \right). \quad (6.1)$$

We now turn to the bundle \mathcal{B} and a description of the semi-flat metric (4.3). We aim to show that the semi-flat metric is just the canonical hyper-Kähler metric on the cotangent bundle of a special Kähler manifold as described in [27, 28]. A clue about how to proceed is in the forms δz_I used in the semi-flat metric. As can be readily checked, it is not true that δz_I is the exterior derivative of z_I . Instead, we have

$$dz_I = \delta z_I + \mathcal{F}_{IJK}^{(3)} (\text{Im } \tau^{-1})^{JL} \text{Im } z_L da^K.$$

We can make sense of this expression using the theory of horizontal lifts. Let X^μ be coordinates on some base manifold M and P_μ the corresponding coordinates on the cotangent bundle obtained by writing a generic 1-form as $\alpha = P_\mu dX^\mu$. Let ∇ be a connection on TM with components $\Theta^\mu_{\nu\rho}$. Then we can define a unique horizontal lift of the frame $\partial_\mu \in TM$ to $T(T^*M)$ by setting

$$D_\mu = \frac{\partial}{\partial X^\mu} + P_\rho \Theta^\rho_{\mu\nu} \frac{\partial}{\partial P_\nu}.$$

⁸View I as a $T\mathcal{M}$ -valued 1-form and act with the exterior covariant derivative d_∇ . We label components of ∇ by $\Theta^\mu_{\nu\rho}$ to avoid confusion with the Levi-Civita connection.

In fact, this can be extended to a frame for $T(T^*M)$ by adjoining the vertical vectors $\partial/\partial P_\mu$, and there is a corresponding dual coframe

$$dX^\mu, \quad \delta P_\mu = dP_\mu - P_\rho \Theta_{\mu\nu}^\rho dX^\nu.$$

Carrying out this construction using the special Kähler connection ∇ with components determined by (6.1), we find the frame

$$D_I = \frac{\partial}{\partial a^I} + \mathcal{F}_{IKL}^{(3)} (\text{Im } \tau^{-1})^{JL} \text{Im } z_J \frac{\partial}{\partial z_K}, \quad \frac{\partial}{\partial z_I} \quad (6.2)$$

for $T(T^*\mathcal{M})$ and coframe

$$da^I, \quad \delta z_I = dz_I - \mathcal{F}_{IKL}^{(3)} (\text{Im } \tau^{-1})^{JL} \text{Im } z_J da^K. \quad (6.3)$$

The key point to notice in this discussion is that the form δz_I defined by horizontal lift exactly coincides with the one appearing in the semi-flat metric (4.3).

Of course we are not done yet since we've not shown that the Coulomb branch \mathcal{B} has anything to do with $T^*\mathcal{M}$, nor have we described the hyper-Kähler structure. To address these issues we use the results of [27, 29]. The moduli space \mathcal{B} has the structure of an algebraic integrable system: in particular, it is a holomorphic symplectic manifold with a fibre bundle structure as described in Section 4 such that the holomorphic symplectic form η vanishes on restriction to the fibres. Furthermore, there is a lattice $\Lambda \cong \mathbb{Z}^r \oplus \tau \mathbb{Z}^r$ (the dual of the electromagnetic charge lattice) such that the fibres are just $\mathcal{J} = \mathbb{C}^r / \Lambda$ and are polarised by Λ^* . Finally, theorem 3.4 of [27] says that such an integrable system is equivalent to the quotient of the cotangent bundle of a special Kähler manifold by a lattice $\Lambda \subset T^*\mathcal{M}$ whose dual is flat with respect to ∇ , such that ∇ has holonomy in the duality group $Sp(2n; \mathbb{Z})$ defined by Λ^* . These are exactly the conditions met by the Coulomb branch of [14] and its associated charge lattice, so we make the identification

$$\mathcal{B} = \frac{T^*\mathcal{M}}{\Lambda} \quad (6.4)$$

with \mathcal{M} the moduli space of the 4d theory.

Describing the hyper-Kähler structure in the large- R limit is now relatively straightforward. For the metric, we use the argument of [27]: given a complex vector space W with

hermitian metric g and dual W^* , there is a canonical ‘hyper-Kähler’ metric G on $W \oplus W^*$ given by

$$G(w_1 \oplus x_1, w_2 \oplus x_2) = g(w_1, w_2) + g^{-1}(x_1, x_2), \quad w_i \in W, \quad x_i \in W^*.$$

In the special Kähler case this can be globalised, since the horizontal lift (6.2) gives an identification⁹

$$T(T^*\mathcal{M}) \cong T\mathcal{M} \oplus T^*\mathcal{M}.$$

Since we already have the well-known metric $\text{Im } \tau$ for \mathcal{M} , we can simply read off the metric on $T^*\mathcal{M}$

$$G = \text{Im } \tau_{IJ} da^I d\bar{a}^J + (\text{Im } \tau^{-1})^{IJ} \delta z_I \delta \bar{z}_J \quad (6.5)$$

where we used δz instead of dz as dictated by horizontal lifting. But this, after some rescalings, is just the semi-flat metric (4.3). We’ve seen that both the full Coulomb branch for the theory on $\mathbb{R}^3 \times S_R^1$ and its hyper-Kähler metric in the large R limit can be constructed canonically from the cotangent bundle of the four dimensional Coulomb branch.

This information is enough to construct the quantum mechanical σ -model on \mathcal{B} , but if we wish to discuss symmetries then we’ll need knowledge of the full hyper-Kähler structure. Fortunately, it is equally as straightforward to read off the Kähler forms and complex structures from the cotangent bundle as it is the metric. Our presentation has a preferred complex structure I^1 with respect to which $\partial/\partial a^I$ and $\partial/\partial z_I$ are holomorphic, and the corresponding Kähler form is

$$\omega_1 = \frac{i}{2} \left(\text{Im } \tau_{IJ} da^I \wedge d\bar{a}^J + (\text{Im } \tau^{-1})^{IJ} \delta z_I \wedge \delta \bar{z}_J \right). \quad (6.6)$$

The other Kähler forms can be read off from the holomorphic symplectic form

$$\eta = \omega_2 + i\omega_3 = da^I \wedge \delta z_I. \quad (6.7)$$

To close this section, we observe that a Kähler potential corresponding to the preferred complex structure is

$$K = \text{Im} \left(\frac{\partial \mathcal{F}}{\partial a^I} \bar{a}^I \right) + 2 (\text{Im } \tau^{-1})^{IJ} \text{Im } z_I \text{Im } z_J. \quad (6.8)$$

⁹Of course, this identification is true for any manifold and any connection ∇ . The special Kähler condition is needed to verify that the Kähler forms on $T^*\mathcal{M}$ are closed.

Note however that this is certainly not a hyper-Kähler potential. Indeed, it was shown in [26] that such an object requires the existence of an isometric action of $SU(2)$ (with extra conditions), which there's no reason to expect our metric (6.5) to exhibit in general.

7 Constructing the σ -Model

We now have all the necessary ingredients to construct the quantum mechanics on \mathcal{B} . The model fits into the general $\mathcal{N} = (1, 1)$ form

$$S = \int dt \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + i g_{\mu\nu} \psi^{\dagger\mu} \frac{D}{dt} \psi^\nu + \frac{1}{4} R_{\mu\nu\rho\sigma} \psi^{\dagger\mu} \psi^\rho \psi^{\dagger\nu} \psi^\sigma \quad (7.1)$$

Here X^μ are generic target space coordinates and ψ^μ are their 1-complex-component fermionic superpartners. The fermion covariant derivative is

$$\frac{D}{dt} \psi^\mu = \dot{\psi}^\mu + \dot{X}^\rho \Gamma_{\rho\nu}^\mu \psi^\nu. \quad (7.2)$$

To formulate this model on \mathcal{B} we need the Levi-Civita connection and curvature associated to the semi-flat metric (6.5) on $T^*\mathcal{M}$. Explicit expressions for these are given in appendix A.

A few words on notation are in order at this point. The expression (7.1) is of course tensorial, so our convention up to now of using the indices I, J, K, \dots for everything is no longer sufficient for bookkeeping purposes: it doesn't distinguish holomorphic/antiholomorphic nor base/fibre indices. The issue is that the index I does not represent a tensorial transformation property, rather a transformation under $Sp(2r; \mathbb{Z})$ duality. The most mathematically respectable way to deal with this would be to use a vielbein-like formalism to relate 'generic' holomorphic coordinates to our special coordinates as in [16]. This will be a little cumbersome for our purposes, so instead we let I, \bar{I} label (anti)holomorphic base directions, I', \bar{I}' label (anti)holomorphic fibre directions and continue to work exclusively with special coordinates. If this is done carefully then no inconsistencies can arise.

The boson kinetic terms are easy to read off from the metric:

$$\mathcal{L}_{\text{Bose}} = \text{Im } \tau_{IJ} \dot{a}^I \dot{a}^J + (\text{Im } \tau^{-1})^{I' \bar{J}'} \frac{\delta z_{I'}}{dt} \frac{\delta \bar{z}_{\bar{J}'}}{dt}, \quad (7.3)$$

where

$$\frac{\delta z_{I'}}{dt} = \dot{z}_{I'} - \mathcal{F}_{I'JK}^{(3)} (\text{Im } \tau^{-1})^{KL} \text{Im } z_L \dot{a}^J$$

reflects the fact that we work in the non-coordinate basis (6.3). Turning now to fermions, we will denote horizontal components by χ^I and vertical components $\zeta_{I'}$. The covariant time derivatives following from (7.2) and (A.1) are

$$\begin{aligned} \frac{D\chi^I}{dt} &= \dot{\chi}^I - \frac{i}{2} (\text{Im } \tau^{-1})^{IL} \mathcal{F}_{JKL}^{(3)} \dot{a}^K \chi^J \\ &\quad + \frac{i}{2} \bar{\mathcal{F}}_{\bar{L}\bar{M}\bar{N}}^{(3)} (\text{Im } \tau^{-1})^{I\bar{L}} (\text{Im } \tau^{-1})^{J'\bar{M}} (\text{Im } \tau^{-1})^{K'\bar{N}} \frac{\delta z_{K'}}{dt} \zeta_{J'} \\ \frac{D\zeta_{I'}}{dt} &= \dot{\zeta}_{I'} + \frac{i}{2} (\text{Im } \tau^{-1})^{J'L} \mathcal{F}_{I'KL}^{(3)} \dot{a}^K \zeta_{J'} \\ &\quad + \frac{i}{2} \mathcal{F}_{I'JL}^{(3)} (\text{Im } \tau^{-1})^{L\bar{K}'} \frac{\delta \bar{z}_{\bar{K}'}}{dt} \chi^J. \end{aligned}$$

The resulting kinetic terms are quite messy, but can be cleared up somewhat by making the redefinition

$$\zeta^I = (\text{Im } \tau^{-1})^{I\bar{J}'} \zeta_{\bar{J}'} \quad (7.5)$$

and using the base Christoffel symbols

$$\Gamma_{JK}^I = -\frac{i}{2} \mathcal{F}_{JKL}^{(3)} (\text{Im } \tau^{-1})^{IL}. \quad (7.6)$$

After making these substitutions we find

$$\begin{aligned} \mathcal{L}_{\text{2-fermi}} &= i \text{Im } \tau_{I\bar{J}} \left[\chi^{\dagger\bar{J}} D_t \chi^I + \zeta^{\dagger\bar{J}} D_t \zeta^I \right] \\ &\quad + i \left[\chi^{\dagger\bar{J}} \zeta^{\bar{M}} + \zeta^{\dagger\bar{J}} \chi^{\bar{M}} \right] \text{Im } \tau_{I\bar{N}} (\text{Im } \tau^{-1})^{K'I} \Gamma_{\bar{J}\bar{M}}^{\bar{N}} \frac{\delta z_{K'}}{dt} + \text{conjugates} \end{aligned} \quad (7.7)$$

where

$$D_t \chi^I = \dot{\chi}^I + \Gamma_{JK}^I \dot{a}^J \chi^K$$

is the base space covariant derivative. We note in passing that χ and ζ appear symmetrically in this expression, which suggests the possibility of combining them into a single object. In fact this will be exactly what we do in Section 8 to exhibit the enhancement of R-symmetry from the $SO(5)$ present in any hyper-Kähler model to the $SO(6) \subset SU(1,1|4)$ required by DLCQ.

As may be seen from the form of the curvature components (A.2), the four-fermion terms fall into two broad classes: contractions with the base space Riemann tensor

$$R_{I\bar{J}K\bar{L}} = -\frac{1}{4} (\text{Im } \tau^{-1})^{M\bar{N}} \mathcal{F}_{IKM}^{(3)} \bar{\mathcal{F}}_{\bar{J}\bar{L}\bar{N}}^{(3)} \quad (7.8)$$

and contractions with the totally symmetric base space tensor

$$\begin{aligned} G_{IJKL} &= -\frac{i}{2} \nabla_I \mathcal{F}_{JKL}^{(3)} \\ &= -\frac{i}{2} \mathcal{F}_{IJKL}^{(4)} + \frac{1}{4} (\text{Im } \tau^{-1})^{MN} \left(\mathcal{F}_{ILM}^{(3)} \mathcal{F}_{JKN}^{(3)} + \mathcal{F}_{JLM}^{(3)} \mathcal{F}_{IKN}^{(3)} + \mathcal{F}_{KLM}^{(3)} \mathcal{F}_{IJN}^{(3)} \right). \end{aligned} \quad (7.9)$$

After using the same redefinition of ζ as for the two-fermion terms (7.5), the latter type gives

$$\mathcal{L}_{4\text{-fermi (a)}} = 2 \text{Re} \left[G_{IJKL} \chi^{\dagger I} \chi^J \zeta^{\dagger K} \zeta^L \right]. \quad (7.10)$$

The Riemann tensor terms are somewhat messier: we find

$$\begin{aligned} \mathcal{L}_{4\text{-fermi (b)}} &= R_{I\bar{J}K\bar{L}} \left[\chi^{\dagger I} \chi^K \chi^{\dagger \bar{J}} \chi^{\bar{L}} + \chi^{\dagger I} \zeta^K \chi^{\dagger \bar{J}} \zeta^{\bar{L}} + \zeta^{\dagger I} \chi^K \zeta^{\dagger \bar{J}} \chi^{\bar{L}} \right. \\ &\quad \left. + \zeta^{\dagger I} \zeta^K \zeta^{\dagger \bar{J}} \zeta^{\bar{L}} + \zeta^{\dagger I} \chi^{\dagger K} \chi^{\bar{J}} \zeta^{\bar{L}} + \chi^{\dagger I} \zeta^K \chi^{\dagger \bar{J}} \chi^{\bar{L}} \right] \end{aligned} \quad (7.11)$$

Although these terms are not especially enlightening at the moment, we will see in Section 8 that they come in exactly the right combinations to admit an extension to $SO(6)$ R-symmetry. The full Lagrangian is the sum of (7.3), (7.7), (7.10) and (7.11).

In identifying the symmetries of our model it will be most convenient to work in the Hamiltonian formalism. This is essentially because the symmetry properties of the objects $\delta z/dt$ are somewhat mysterious and will become much clearer after Legendre transform. To that end, we begin by computing the canonical momenta

$$\begin{aligned} P_I &= \frac{\partial \mathcal{L}}{\partial \dot{a}^I} = \text{Im } \tau_{I\bar{J}} \dot{a}^{\bar{J}} + 2i \text{Im } z_K \frac{\delta \bar{z}_{\bar{J}'}}{dt} \frac{\partial}{\partial a^I} (\text{Im } \tau^{-1})^{\bar{J}'K} \\ &\quad + \frac{1}{2} \text{Im } \tau_{K\bar{J}} (\text{Im } \tau^{-1})^{KM} \mathcal{F}_{ILM}^{(3)} \left(\chi^{\dagger \bar{J}} \chi^L + \zeta^{\dagger \bar{J}} \zeta^L \right) \\ &\quad - 2R_{I\bar{J}K\bar{L}} (\text{Im } \tau^{-1})^{KM} \text{Im } z_M \left(\chi^{\dagger \bar{L}} \zeta^{\bar{J}} + \zeta^{\dagger \bar{M}} \chi^{\bar{J}} \right) \\ P^{I'} &= \frac{\partial \mathcal{L}}{\partial \dot{z}_{I'}} = (\text{Im } \tau^{-1})^{I'\bar{J}'} \frac{\delta \bar{z}_{\bar{J}'}}{dt} \\ &\quad - \frac{1}{2} (\text{Im } \tau^{-1})^{I'\bar{L}} \bar{\mathcal{F}}_{\bar{J}\bar{K}\bar{L}}^{(3)} \left(\chi^{\dagger \bar{K}} \zeta^{\bar{J}} + \zeta^{\dagger \bar{K}} \chi^{\bar{J}} \right). \end{aligned} \quad (7.12)$$

Following [6, 10] we also define the covariant momenta

$$\Pi_I = \text{Im } \tau_{I\bar{J}} \dot{a}^{\bar{J}}, \quad \Pi^{I'} = (\text{Im } \tau^{-1})^{I'\bar{J}'} \frac{\delta \bar{z}_{\bar{J}'}}{dt} \quad (7.13)$$

in terms of which the Hamiltonian is simply

$$H = (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \text{Im } \tau_{I'\bar{J}'} \Pi^{I'} \bar{\Pi}^{\bar{J}'} - \mathcal{L}_{4\text{-fermi}}. \quad (7.14)$$

The commutation relations of these objects are subtle and require the Dirac bracket procedure to get right, the details of which we omit. We obtain the non-vanishing commutators:

$$\begin{aligned} [a^I, \Pi_J] &= i\delta_J^I \\ [z_{I'}, \Pi_J] &= i \text{Im } z_K \mathcal{F}_{I'JL}^{(3)} (\text{Im } \tau^{-1})^{KL} & [z_{I'}, P^{J'}] &= i\delta_{I'}^{J'} \\ [\Pi_I, P^{J'}] &= \frac{1}{2} \mathcal{F}_{IK'L}^{(3)} (\text{Im } \tau^{-1})^{J'L} P^{K'} & [\Pi_I, P^{\bar{J}'}] &= -\frac{1}{2} \mathcal{F}_{IK'L}^{(3)} (\text{Im } \tau^{-1})^{\bar{J}'L} P^{K'} \\ \{\chi^I, \chi^{\dagger\bar{J}}\} &= (\text{Im } \tau^{-1})^{I\bar{J}} & \{\zeta^I, \zeta^{\dagger\bar{J}}\} &= (\text{Im } \tau^{-1})^{I\bar{J}} \\ [\Pi_I, \chi^J] &= i\Gamma_{IK}^J \chi^K & [\Pi_I, \chi^{\dagger J}] &= i\Gamma_{IK}^J \chi^{\dagger K} \\ [\Pi_I, \zeta^J] &= i\Gamma_{IK}^J \zeta^K & [\Pi_I, \zeta^{\dagger J}] &= i\Gamma_{IK}^J \zeta^{\dagger K} \end{aligned}$$

It is important to notice that the commutation relations of $P^{I'}$ are consistent with zero, so we can truncate to the sector of zero momentum around the fibres. Of course such a truncation is the natural one to consider in the $R \rightarrow \infty$ limit of the moduli space, in which the torus fibres become small [14]. In the following sections we will see that this is crucial to revealing the superconformal symmetry of our model.

8 R-Symmetry Enhancement $SO(5) \rightarrow SO(6)$

We can now begin our analysis of the symmetries of our model. As reviewed in Section 5, a hyper-Kähler σ -model must have $\mathcal{N} = (4, 4)$ supersymmetry with $SO(5)$ R-symmetry acting purely on fermions. In fact we will see that in this case the R-symmetry extends to $SO(6)$, but it will be a good first step to put the Hamiltonian into manifestly $SO(5)$ -invariant form and construct the generators and supersymmetries.¹⁰

¹⁰We use A, B to index the $\mathbf{4}$ of $SO(6)$ and \bar{A}, \bar{B} the $\bar{\mathbf{4}}$. Indices are raised/lowered with $\delta_{AB} = \text{diag}(1, 1, 1, 1)$. We keep the $\mathbf{4}$ and $\bar{\mathbf{4}}$ separate even in $SO(5)$, in view of the forthcoming extension to $SO(6)$. $SO(5)$ has antisymmetric invariant tensor Ω_{AB} and where necessary we take $\Omega_{23} = \Omega_{41} = 1$.

The $SO(5)$ R-symmetry generators are as given in [10]: in a notation emphasising the $SU(2)$ subgroups associated to each complex structure I^a they are

$$\begin{aligned} J_+^a &= \frac{1}{2} \omega_{\mu\nu}^a \psi^{\dagger\mu} \psi^{\dagger\nu} & J_-^a &= \frac{1}{2} \omega_{\mu\nu}^a \psi^\nu \psi^\mu \\ R^a &= -\frac{i}{2} \omega_{\mu\nu}^a \psi^{\dagger\mu} \psi^\nu & J_3 &= \frac{1}{2} (g_{\mu\nu} \psi^{\dagger\mu} \psi^\nu - 2r). \end{aligned} \quad (8.1)$$

In terms of the canonical quantisation in which wavefunctions with fermion number F become differential forms of degree F , J_+^a is wedging with the Kähler form ω^a , R^a is the action of I^a and J_3 counts degree. We can easily read off explicit expressions for these generators from (6.6) and (6.7) using the rules

$$da^I \leftrightarrow \chi^{\dagger I}, \quad (\text{Im } \tau^{-1})^{J\bar{I}'} \delta \bar{z}_{\bar{I}'} \leftrightarrow \zeta^{\dagger J}.$$

The detailed coefficients are not important, but notice that all generators follow the pattern

$$T \sim \text{Im } \tau \times \text{holomorphic fermion} \times \text{antiholomorphic fermion}.$$

This means that the (anti)holomorphic fermions carry separate actions of $SO(5)$. Indeed, if we define the objects

$$\psi^{IA} = (\chi^I, \chi^{\dagger I}, \zeta^I, \zeta^{\dagger I})$$

then we see that ψ^{IA} transforms in the $\mathbf{4}$ and $\bar{\psi}^{\bar{I}\bar{A}} = (\psi^{IA})^\dagger$ in the $\bar{\mathbf{4}}$. They satisfy the simple anticommutation relations

$$\left\{ \psi^{IA}, \bar{\psi}^{\bar{J}\bar{B}} \right\} = \delta^{A\bar{B}} (\text{Im } \tau^{-1})^{I\bar{J}}.$$

We now put the Hamiltonian (7.14) into a manifestly $SO(5)$ -invariant form. To begin with note that we have

$$[\Pi_I, \psi^{JA}] = i\Gamma_{IK}^J \psi^{KA}, \quad [\Pi_I, \bar{\psi}^{\bar{J}\bar{A}}] = 0$$

so that Π_I must be $SO(5)$ -neutral and the term $(\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}}$ in the Hamiltonian is $SO(5)$ -invariant. Another straightforward part is the chiral term

$$2 \text{Re} (G_{IJKL} \chi^{\dagger I} \chi^J \zeta^{\dagger K} \zeta^L)$$

which may be written as

$$H_{\text{chiral}} = \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \quad (8.2)$$

using the symmetry of G_{IJKL} .

The remaining terms are less obvious. It will prove convenient to work in terms of the canonical momentum $P^{I'}$ rather than its covariant form $\Pi^{I'}$, so that

$$\begin{aligned} \text{Im } \tau_{I'\bar{J}'} \Pi^{I'} \bar{\Pi}^{\bar{J}'} &= \text{Im } \tau_{I'\bar{J}'} P^{I'} \bar{P}^{\bar{J}'} + \text{Re} \left[\mathcal{F}_{I'JK}^{(3)} (\chi^{\dagger J} \zeta^K + \zeta^{\dagger J} \chi^K) P^{I'} \right] \\ &+ R_{I\bar{J}K\bar{L}} (\chi^{\dagger I} \zeta^K + \zeta^{\dagger I} \chi^K) (\chi^{\dagger \bar{J}} \zeta^{\bar{L}} + \zeta^{\dagger \bar{J}} \chi^{\bar{L}}). \end{aligned} \quad (8.3)$$

The first term on the right is manifestly $SO(5)$ -invariant and the second can be put in the form

$$\text{Re} \left[\mathcal{F}_{I'JK}^{(3)} (\chi^{\dagger J} \zeta^K + \zeta^{\dagger J} \chi^K) P^{I'} \right] = \frac{1}{2} \text{Re} \left(\mathcal{F}_{I'JK}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} P^{I'} \right).$$

We emphasise for later the use of the $SO(5)$ symplectic form Ω_{AB} which will clearly obstruct any possible extension to $SO(6)$. The four-fermion terms in (8.3) can be combined with the remainder of the Hamiltonian (7.11) to obtain

$$H_{\text{Riemann}} = \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}^{\bar{J}} \psi^{KB} \bar{\psi}^{\bar{L}}. \quad (8.4)$$

Taking everything together, we have the $SO(5)$ invariant Hamiltonian

$$\begin{aligned} H &= (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \\ &+ \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}^{\bar{J}} \psi^{KB} \bar{\psi}^{\bar{L}} \\ &+ \text{Im } \tau_{I'\bar{J}'} P^{I'} \bar{P}^{\bar{J}'} + \frac{1}{2} \text{Re} \left(\mathcal{F}_{I'JK}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} P^{I'} \right) \end{aligned} \quad (8.5)$$

We can also put the supercharges into $SO(5)$ multiplets. In a generic hyper-Kähler σ -model of the form (7.1) these charges are (see e.g [10])

$$Q = i\psi^{\dagger\mu} \Pi_{\mu} \quad \quad Q^a = -i\psi^{\dagger\mu} I_{\mu}^{a\nu} \Pi_{\nu}$$

where we were not careful about operator ordering¹¹. Using the complex structures (6.7) we can easily read off the charges:

$$Q = i\chi^{\dagger I} \Pi_I + i \text{Im } \tau_{I'\bar{J}'} \zeta^{\dagger \bar{J}'} P^{I'} + \frac{i}{2} \zeta^{\dagger L} \mathcal{F}_{JLM}^{(3)} (\chi^{\dagger M} \zeta^J + \zeta^{\dagger M} \chi^J) - \text{complex conjugate},$$

¹¹In [10] we used $Q^{\dagger} = -i\Pi_{\mu} \psi^{\mu}$ in order to ensure the validity of the exterior algebra representation $Q \rightarrow d, Q^{\dagger} \rightarrow d^{\dagger}$. In this paper, we use a different ordering convention to make the $SO(5)$ invariance manifest. Strictly speaking, much of what follows is only valid at the level of Poisson brackets, but we do not anticipate this causing any problems

with similar expressions for Q^a . Taking suitable linear combinations of these leads to expressions which manifestly transform in the $\mathbf{4}$ of $SO(5)$,

$$Q^A = \psi^{IA} \Pi_I + \frac{1}{12} \epsilon^A_{\bar{B}\bar{C}\bar{D}} \bar{\mathcal{F}}_{\bar{I}\bar{J}\bar{K}}^{(3)} \bar{\psi}^{\bar{I}\bar{B}} \bar{\psi}^{\bar{J}\bar{C}} \bar{\psi}^{\bar{K}\bar{D}} + \text{Im } \tau_{I'\bar{J}} P^{I'} \Omega^A_{\bar{B}} \bar{\psi}^{\bar{J}\bar{B}} \quad (8.6)$$

along with the conjugate $\bar{Q}^{\bar{A}} = (Q^A)^\dagger$ which transforms in the $\bar{\mathbf{4}}$. These charges obey the standard supersymmetry algebra

$$\begin{aligned} \{Q^A, Q^B\} &= 0 \\ \{Q^A, \bar{Q}^{\bar{B}}\} &= \delta^{A\bar{B}} H. \end{aligned}$$

As remarked briefly above, it is clear from the $SO(5)$ -manifest form of both the Hamiltonian and the supercharges that $SO(5)$ is the largest symmetry we can get without changing something, since the expressions (8.5) and (8.6) both require the $SO(5)$ -invariant tensor Ω_{AB} which does not exist in $SO(6)$. Furthermore we do not expect conformal invariance without some modification, as the torus fibre has a fixed finite size. In the following we tackle each extension in turn, and show that they can both be achieved via the same truncation to the sector of zero fibre momentum.

We can extend the $SO(5)$ R-symmetry (8.1) to $SO(6) \simeq SU(4)$ via the obvious generalisation

$$R^{A\bar{B}} = i \text{Im } \tau_{I\bar{J}} \left(\psi^{IA} \bar{\psi}^{\bar{J}\bar{B}} - \frac{1}{4} \delta^{A\bar{B}} \psi^{IC} \bar{\psi}^{\bar{J}\bar{C}} \right), \quad (8.8)$$

where the second term removes a trace part and reduces $U(4) \rightarrow SU(4)$. These obey the expected commutation relations

$$\begin{aligned} [R^{A\bar{B}}, R^{C\bar{D}}] &= i \left(\delta^{C\bar{B}} R^{A\bar{D}} - \delta^{A\bar{D}} R^{C\bar{B}} \right) \\ [R^{A\bar{B}}, \psi^{IC}] &= i \left(\delta^{C\bar{B}} \psi^{IA} - \frac{1}{4} \delta^{A\bar{B}} \psi^{IC} \right) \\ [R^{A\bar{B}}, \bar{\psi}^{\bar{I}\bar{C}}] &= -i \left(\delta^{A\bar{C}} \bar{\psi}^{\bar{I}\bar{B}} - \frac{1}{4} \delta^{A\bar{B}} \bar{\psi}^{\bar{I}\bar{C}} \right) \\ [R^{A\bar{B}}, \Pi_I] &= 0 = [R^{A\bar{B}}, P^{I'}], \end{aligned}$$

which confirm that ψ transforms in the $\mathbf{4}$, $\bar{\psi}$ in the $\bar{\mathbf{4}}$ and that $\Pi_I, P^{I'}$ are neutral.

This is enough to demonstrate that the majority of the terms in both the Hamiltonian (8.5) and the supercharges (8.6) have the correct charges under $SO(6)$. The problematic terms are of course those relying on the $SO(5)$ -invariant form Ω_{AB} , but we note that they always appear multiplying the fibre momentum $P^{I'}$. Recall that we are considering the compactification on $\mathbb{R}^3 \times S_R^1$ in the limit $R \rightarrow \infty$ where the torus fibres \mathcal{J} become small. It is then natural to truncate to zero fibre momentum, since we can Fourier expand around the fibres¹² and see that states with nonzero $P^{I'}$ have divergent energy. We conclude that our model admits $SO(6)$ R-symmetry at large R as required by DLCQ.

9 $SU(1, 1|4)$ and Scale-Invariant Special Kähler Geometry

Finally we turn to superconformal invariance. Recall that $SU(1, 1|4)$ is a simple supergroup with bosonic part

$$SO(2, 1) \times U(4)$$

and a total of 16 fermions: (Q^A, S^B) transform in the $(\mathbf{2}, \mathbf{4})$ and $(\bar{Q}^{\bar{A}}, \bar{S}^{\bar{B}})$ in the $(\mathbf{2}, \bar{\mathbf{4}})$.

As reviewed in Section 5, superconformal invariance requires the target space to admit a homothety, that is a vector D satisfying $\mathcal{L}_D g = 2g$, which acts as the dilatation operator. There is no reason why a generic special Kähler manifold might be expected to admit such an object, so in order to proceed we need to make a definition. We call a geometry *scale-invariant special Kähler* (SISK) if there is a prepotential satisfying the further condition

$$a^I \frac{\partial}{\partial a^I} \mathcal{F} = 2\mathcal{F}. \quad (9.1)$$

If the prepotential is of this form then it is clear that the Coulomb branch of the 4d theory has a homothety

$$D = a^I \frac{\partial}{\partial a^I} + \bar{a}^I \frac{\partial}{\partial \bar{a}^I}. \quad (9.2)$$

One might wonder whether the SISK condition has any interesting solutions. A trivial one has \mathcal{F} a quadratic polynomial in the a^I , corresponding to a flat manifold. This is of some limited physical interest as of course it corresponds to the finite $\mathcal{N} = 4$ theory and to

¹²This is true at least on a locally trivial patch of the fibre bundle.

the diagonal $U(1)$ in our quiver model (3.1), but we'd like to do better. The SISK condition follows if and only if \mathcal{F} is homogeneous of degree 2, so any function of the form

$$\mathcal{F} = (a^1)^2 f\left(\frac{a^I}{a^J}\right)$$

for arbitrary holomorphic f will do. There is a large family of such prepotentials available in physics. It is perhaps no surprise that they arise from Coulomb branches of $\mathcal{N} = 2$ superconformal theories in four dimensions, whose microscopic scale invariance is reflected in a scale-invariant metric in the low-energy theory. In particular, of course, our quiver model of DLCQ is of this form.

Returning to the construction of $SU(1,1|4)$, we try to give an expression for the special conformal generator K . As reviewed in Section 5, this must be given by the Kähler potential, but the question here is which one? (6.8) is a possible potential on the total space \mathcal{B} but only with respect to the preferred complex structure, whereas [10] suggests it must be a hyper-Kähler potential. Fortunately the same truncation to z -independent functions used in Section 8 comes to the rescue, and it will turn out to be sufficient to use the base space Kähler potential

$$K = \text{Im} \left(\frac{\partial \mathcal{F}}{\partial a^I} \bar{a}^I \right) \quad (9.3)$$

obtained from (6.8) by setting $z = 0$.

With special conformal generator as above, it is straightforward to calculate the dilatation operator using the rule $[H, K] = -iD$, and we find (note that we always assume the SISK condition from now on)

$$D = a^I \Pi_I + \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}} \quad (9.4)$$

in agreement with the homothety (9.2). Notice in particular that the bosons $a^I, \bar{a}^{\bar{I}}$ have dimension 1

$$[D, a^I] = -ia^I$$

while, as a consequence of the SISK condition, all fermions have dimension 0. In fact after we make the truncation to $P^{I'} = 0$, so that

$$\begin{aligned} H &= (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \\ &+ \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}^{\bar{J}} \psi^{KB} \bar{\psi}^{\bar{L}}, \end{aligned} \quad (9.5)$$

we find that the full $SO(2, 1)$ conformal algebra

$$[D, H] = 2iH, \quad [D, K] = -2iK, \quad [H, K] = -iD$$

is obeyed. It is also manifest that $SO(2, 1)$ commutes with the $SO(6)$ R-symmetry (8.8). The extra terms present in H for $P^{I'} \neq 0$ break the relation $[D, H] = 2iH$ as $P^{I'}$ has the wrong dimension. Explicit expressions for the ‘deformed’ algebra occurring for $P^{I'} \neq 0$ are given in appendix C.

Making the same truncation for the supercharges, so they read

$$Q^A = \psi^{IA} \Pi_I + \frac{1}{12} \epsilon^A_{\bar{B}\bar{C}\bar{D}} \bar{\mathcal{F}}_{\bar{I}\bar{J}\bar{K}}^{(3)} \bar{\psi}^{\bar{I}\bar{B}} \bar{\psi}^{\bar{J}\bar{C}} \bar{\psi}^{\bar{K}\bar{D}}, \quad (9.6)$$

we find

$$[D, Q^A] = iQ^A$$

as required. Again the extra $P^{I'} \neq 0$ terms in (8.6) have the wrong dimension. We can also define the superconformal generators S by the rule $[K, Q^A] = iS^A$, giving

$$S^A = \text{Im } \tau_{IJ} \bar{a}^{\bar{J}} \psi^{IA} \quad (9.7)$$

along with their conjugates $\bar{S}^{\bar{A}}$. These generators have the correct $SO(6)$ transformation properties and dimensions, as well as obeying the expected relations

$$\begin{aligned} \{S^A, S^B\} &= 0 & [K, S^A] &= 0 \\ \{S^A, \bar{S}^{\bar{B}}\} &= \delta^{A\bar{B}} K & [H, S^A] &= -iQ^A, \end{aligned}$$

the last relation also being broken by the $P^{I'} \neq 0$ terms in H .

It remains to check the $\{Q, S\}$ relations. Doing so reveals a $U(1)$ R-symmetry

$$\mathcal{R} = i \left(a^I \Pi_I - \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}} \right) + \frac{1}{2} \text{Im } \tau_{IJ} \psi^{IA} \bar{\psi}^{\bar{J}}_A \quad (9.8)$$

with charges

a^I	$\bar{a}^{\bar{I}}$	ψ^{IA}	$\bar{\psi}^{\bar{I}\bar{A}}$	Q^A	S^A	$\bar{Q}^{\bar{A}}$	$\bar{S}^{\bar{A}}$
1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

With this definition we have

$$\begin{aligned} \{Q^A, S^B\} &= 0 \\ \{Q^A, \bar{S}^{\bar{B}}\} &= \frac{1}{2} \delta^{A\bar{B}} (D - i\mathcal{R}) - R^{A\bar{B}}, \end{aligned}$$

where the first relation is also broken when $P^{I'} \neq 0$. This completes our construction of $SU(1, 1|4)$: for convenience, we collect the operators and commutation relations in appendix B.

10 Reduction from Four Dimensions

In this section we would like to present a simpler perspective on the results given above. In the limit where we restrict to states with zero momentum along the fibre, the model reduces to a σ -model with the special Kähler base as target. In fact (9.5) is the Hamiltonian for a novel type of quantum mechanical σ -model with special Kähler target. It can also be thought of as quantum mechanics on the Coulomb branch of a four-dimensional gauge theory with $\mathcal{N} = 2$ superconformal symmetry. From this point of view, it is natural to suspect that we could have derived it directly from the low-energy effective action of the 4d theory by dimensional reduction to quantum mechanics. We will now perform the reduction and discuss the symmetries of the resulting σ -model in this context.

We will begin by considering an arbitrary $\mathcal{N} = 2$ supersymmetric field theory in four dimensions with gauge group of rank r . The bosonic symmetry of the model includes the Lorentz group $SO(3, 1) \simeq SL(2)_A \times SL(2)_B$ as well as an $SU(2)$ R-symmetry. If the theory is superconformal, it will also have a non-anomalous $U(1)$ R-symmetry. The supercharges Q_α^i and $\bar{Q}_{\dot{\alpha}i}$ transform in the $(\mathbf{2}, \mathbf{1}, \mathbf{2})_{+1} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1}$ of the global symmetry group

$$G_{\text{global}} = SL(2)_A \times SL(2)_B \times SU(2)_R \times U(1)_R.$$

The theory has a Coulomb branch where the scalars in the vector multiplet acquire non-zero expectation values and the gauge group is broken to $U(1)^r$ by the adjoint Higgs mechanism. The massless fields consist of r $U(1)$ vector multiplets with complex scalars a^I as lowest components, where $I = 1, 2, \dots, r$ labels the Cartan subalgebra of the gauge group. The supersymmetry multiplet combines a^I with left-handed Weyl fermions $\lambda_\alpha^I, \psi_\alpha^I$ and the self-dual part of the $U(1)$ gauge field strength, $(v^{\text{SD}})_{mn}^I$. The charge conjugate multiplet has lowest component \bar{a}^I and also includes right-handed Weyl fermions $\bar{\lambda}_{\dot{\alpha}}^I, \bar{\psi}_{\dot{\alpha}}^I$ together with the anti-self-dual part of the $U(1)$ gauge field strength, $(v^{\text{ASD}})_{mn}^I$.

The scalars a^I and \bar{a}^I parameterise a vacuum moduli space which is a special Kähler manifold of complex dimension r . The metric is determined in terms of the holomorphic prepotential $\mathcal{F}(a)$ by (4.2). We also use the Christoffel symbols (7.6) to define a covariant derivative for the fermions:

$$D_m \psi_\alpha^I = \partial_m \psi_\alpha^I + \Gamma_{JK}^I \partial_m a^J \psi_\alpha^K.$$

The full low-energy effective Lagrangian is (see e.g [30])

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

where (up to an irrelevant overall factor of $1/4\pi$)

$$\begin{aligned} \mathcal{L}_1 &= -\text{Im } \tau_{IJ} \left[\partial_m a^I \partial^m \bar{a}^J + i \bar{\psi}_\alpha^J (\bar{\sigma}^m)^{\dot{\alpha}\alpha} D_m \psi_\alpha^I + i \bar{\lambda}_\alpha^J (\bar{\sigma}^m)^{\dot{\alpha}\alpha} D_m \lambda_\alpha^I \right] \\ \mathcal{L}_2 &= \text{Im} \left[-\frac{1}{2} \mathcal{F}_{IJ}^{(2)} (v^{\text{SD}})^I_{mn} (v^{\text{SD}})^{Jmn} + \right. \\ &\quad \left. \frac{1}{\sqrt{2}} \mathcal{F}_{IJK}^{(3)} \lambda^{\alpha I} (\sigma^{mn})_\alpha^\beta \psi_\beta^J v_{mn}^K + \frac{1}{4} \mathcal{F}_{IJKL}^{(4)} \psi^{\alpha I} \psi_\alpha^J \lambda^{\beta K} \lambda_\beta^L \right] \\ \mathcal{L}_3 &= -\text{Im } \tau_{IJ} \left[F^I \bar{F}^J + \frac{1}{2} D^I D^J \right] \\ &\quad - \frac{1}{2} \text{Im} \left[\mathcal{F}_{IJK}^{(3)} \left(\bar{F}^I (\psi^{\alpha J} \psi_\alpha^K + \lambda^{\alpha J} \lambda_\alpha^K) - i\sqrt{2} D^I \psi^{\alpha J} \lambda_\alpha^K \right) \right]. \end{aligned}$$

Here we have introduced complex and real auxiliary fields F^I and D^I respectively which can be eliminated using their equations of motion. The supersymmetry transformations for this action can be found in [30].

We now consider the dimensional reduction of this action to $0+1$ dimensions by setting

$$\begin{aligned} \partial_m &= \frac{\partial}{\partial t} & m &= 0 \\ &= 0 & m &= 1, 2, 3. \end{aligned}$$

The surviving fields are the scalars a^I , \bar{a}^I , the fermions ψ_α^I , λ_α^I , $\bar{\psi}_\alpha^I$, $\bar{\lambda}_\alpha^I$, the auxiliary fields F^I , \bar{F}^I , D^I and the electric field strength $E_l^I = v_{0l}^I$ for $l = 1, 2, 3$.

The reduction breaks the four-dimensional Lorentz group down to three-dimensional spatial rotations denoted $SU(2)_N$ and the reduced theory has a manifest bosonic symmetry group

$$SU(2)_N \times SU(2)_R \times U(1)_R.$$

Less obviously the fields of the reduced theory can be combined into multiplets of an $SU(4)_R$ which contains $SU(2)_N \times SU(2)_R$ as a subgroup. The $SU(2)_N \times SU(2)_R$ quantum numbers of the various surviving fields and their lift to $SU(4)_R$ are given in the table below:

	$SU(2)_N$	$SU(2)_R$	$SU(4)_R$
a, \bar{a}	1	1	1
ψ, λ	2	2	4
$\bar{\psi}, \bar{\lambda}$	2	2	$\bar{4}$
E	3	1	6
F, \bar{F}, D	1	3	

As indicated in the table the fermion components can easily be assembled into a **4** and **$\bar{4}$** of $SU(4)_R$. The electric field strength E and auxiliary fields are combined to form a bosonic field $\vec{\chi}$ transforming in the **6** of $SU(4)_R \simeq SO(6)_R$. Explicitly we form an $SO(6)$ vector

$$\vec{\chi} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ D \\ \sqrt{2}\text{Re}[F] \\ \sqrt{2}\text{Im}[F] \end{pmatrix}.$$

The vector **6** of $SO(6)$ corresponds to a second rank pseudo-real antisymmetric tensor representation of $SU(4)$. The map between these representations involves a vector $\vec{\Sigma}_{AB}$ of 4×4 anti-symmetric matrices. One possible choice is

$$\vec{\Sigma} = (\eta^1, \eta^2, \eta^3, i\bar{\eta}^1, i\bar{\eta}^2, i\bar{\eta}^3),$$

where η_{AB}^a and $\bar{\eta}_{AB}^a$ are the 't Hooft symbols corresponding to self-dual and anti-self-dual generators of $SO(4)$ respectively. Thus we define a complex anti-symmetric tensor field $\chi_{AB} = \vec{\chi} \cdot \vec{\Sigma}_{AB}$ which obeys the pseudo-reality condition

$$\bar{\chi}^{AB} = \frac{1}{2}\epsilon^{ABCD}\chi_{CD}.$$

In summary, we now introduce new fields

$$\begin{aligned} a^I, & \quad \psi^{IA}, & \chi_{AB}^I \\ \bar{a}^I, & \quad \bar{\psi}^{I\bar{A}}, & \bar{\chi}_{\bar{A}\bar{B}}^I \end{aligned}$$

in the $(\mathbf{1} \oplus \mathbf{4} \oplus \mathbf{6}) \oplus (\mathbf{1} \oplus \bar{\mathbf{4}} \oplus \mathbf{6})$ of $SU(4)_R$.

The Lagrangian of the reduced theory can be written in a manifestly $SU(4)_R$ -invariant form

$$\begin{aligned}\mathcal{L} = & \text{Im } \tau_{IJ} \left[\dot{a}^I \dot{a}^J + i \bar{\psi}_A^J D_t \psi^{IA} + \chi_{AB}^I \bar{\chi}^{JAB} \right] \\ & + \frac{1}{4\pi} \text{Im} \left[\frac{1}{\sqrt{2}} \mathcal{F}_{IJK}^{(3)} \chi_{AB}^I \psi^{JA} \psi^{KB} + \frac{1}{48i} \mathcal{F}_{IJKL}^{(4)} \epsilon_{ABCD} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD} \right].\end{aligned}$$

In the above the time derivatives of χ_{AB} do not appear and it can be treated as an auxiliary field. This may seem a little odd as three of the six independent components of χ_{AB} started life as electric field strengths in four dimensions and are naturally thought of as time derivatives of a vector potential. However, this is consistent after our dimensional reduction where corresponding spatial derivatives of the vector potential are set to zero. Finally to make contact with theory of the previous section, we integrate out the auxiliary fields to get the following Lagrangian

$$\begin{aligned}\mathcal{L} = & \text{Im } \tau_{IJ} \dot{a}^I \dot{a}^J + i \text{Im } \tau_{IJ} \bar{\psi}_A^J D_t \psi^{IA} \\ & - \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) - \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}_A^{\bar{J}} \psi^{KB} \bar{\psi}_B^{\bar{L}},\end{aligned}$$

where $R_{I\bar{J}K\bar{L}}$ and G_{IJKL} are as in (7.8) and (7.9). Performing a Legendre transform on the above Lagrangian, we arrive at the $SU(4) \simeq SO(6)$ -invariant Hamiltonian (9.5) of the previous section.

11 Discussion

In this paper we have shown that a class of quantum mechanical σ -models with scale-invariant special Kähler target space have $SU(1, 1|4)$ superconformal invariance. The Coulomb branches of four-dimensional superconformal field theories with $\mathcal{N} = 2$ supersymmetry provide a large class of examples. We also study the related quantum mechanical σ -model on the Coulomb branch of the same $\mathcal{N} = 2$ theories compactified down to three dimensions on a circle of radius R . The DLCQ description of the $\mathcal{N} = 4$ theory arises as a special case corresponding to the four-dimensional \hat{A}_{N-1} quiver described in Section 3. Each of these models is determined by the same data as the corresponding Seiberg-Witten solution, namely a complex curve Σ with holomorphic differential λ . These in turn specify the holomorphic prepotential \mathcal{F} , which determines the Hamiltonian and the other superconformal symmetry generators explicitly. The simplest possible model one could consider corresponds to the classical prepotential for

a simple gauge group of rank r

$$\mathcal{F} = \sum_{I=1}^r \frac{1}{2} \tau a_I^2.$$

In this case at least, the target space has only orbifold singularities coming from the fixed points of the Weyl group and the superconformal symmetry generators are globally defined.

The situation becomes more interesting as soon as quantum corrections to the Coulomb branch metric are included. These famously lead to singular submanifolds on which charged BPS states of the four-dimensional $\mathcal{N} = 2$ theory become massless¹³. For a model of rank one, in suitable local coordinates, the prepotential has the characteristic form

$$\mathcal{F} \sim a^2 \log a.$$

This causes at least two problems. First, the corresponding target space metric is singular at $a = 0$. This means that quantum mechanics on the manifold is not obviously well-defined, at least for states with wavefunctions supported near the singularity. It is therefore necessary to regulate the model by resolving the singularity. In the present case a natural regulator is obtained by working with the full hyper-Kähler model of Section 7 at a finite value of the compactification radius R . Although the semi-flat metric (4.3) we have studied in this paper is also singular, it is known [15] that the logarithmic singularities discussed above are resolved by instanton corrections¹⁴ for any finite value of R . Of course the same instanton effects will also break the superconformal invariance of the model explicitly.

The second, related, problem introduced by the logarithmic singularity discussed above is that the superconformal generators themselves are no longer single-valued on the Coulomb branch. The special conformal generator K , in particular, depends on the first derivative of the Kähler potential which has non-trivial monodromy around the singular point. This

¹³A related phenomenon in the quantum mechanics model is that the corresponding cycles of the Jacobian torus decompactify at these points and states with momentum in the fibre directions fail to decouple as $R \rightarrow \infty$.

¹⁴In the superconformal models of interest here, singularities of higher codimension persist even at finite R . In the Higgs branch description, these are essentially the familiar singularities associated with small instantons on \mathbb{R}^4 . The standard approach to their resolution involves the introduction of spacetime non-commutativity [31].

means that we can only define the action of the superconformal group locally in some patch which does not contain a singular point. Alternatively one could consider globally defined generators acting on an infinitely-branched cover of the target space. These issues and their implications for the DLCQ of the $\mathcal{N} = 4$ theory will be revisited in a forthcoming paper [7].

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A Connection and curvature of $T^*\mathcal{M}$

In this appendix we give the connection and curvature forms for the Levi-Civita connection corresponding to the metric (6.5). We work in the bases (6.2) and (6.3) determined by horizontal lifting and use the notational conventions outlined at the beginning of section 7. The connection components are defined by

$$\nabla_{e_A} e_B = \theta_{BA}^C e_C$$

where e_A is a generic frame vector, and the nonzero components are

$$\begin{aligned} \theta_{JI}^K &= -\frac{i}{2} (\text{Im } \tau^{-1})^{KL} \mathcal{F}_{IJL}^{(3)} \\ \theta^{KJ'I'} &= \frac{i}{2} \mathcal{F}_{LMN}^{(3)} (\text{Im } \tau^{-1})^{KL} (\text{Im } \tau^{-1})^{J'M} (\text{Im } \tau^{-1})^{I'N} \\ \theta_{K'J}^{\bar{I}'} &= \frac{i}{2} \mathcal{F}_{JK'L}^{(3)} (\text{Im } \tau^{-1})^{L\bar{I}'} \\ \theta_{K'I}^{J'} &= \frac{i}{2} \mathcal{F}_{IK'L}^{(3)} (\text{Im } \tau^{-1})^{J'L}. \end{aligned} \tag{A.1}$$

The fact that the components with mixed (anti)holomorphic indices do not vanish is perfectly consistent since the frame vectors D_I are not holomorphic.

The curvature forms are defined by

$$\Omega_B^A = d\theta_B^A + \theta_C^A \wedge \theta_B^C.$$

To construct the action (7.1) we need the curvature tensor with all lowered indices. The nonzero components of this are

$$\begin{aligned}
\Omega_{\bar{I}JK\bar{L}} &= R_{\bar{I}JK\bar{L}} & \Omega_{JK}^{\bar{I}'}{}^{\bar{L}'} &= -(\text{Im } \tau^{-1})^{\bar{I}'N} (\text{Im } \tau^{-1})^{\bar{L}'M} G_{JKMN} \\
\Omega_{\bar{I}J}{}^{K'\bar{L}'} &= R_{\bar{I}J}{}^{K'\bar{L}'} & \Omega_{J\bar{K}}^{\bar{I}'}{}^{L'} &= R_{\bar{K}}^{\bar{I}'}{}^{L'}{}_J \\
\Omega_{\bar{I}'}{}^{J'K'\bar{L}'} &= R_{\bar{I}'}{}^{J'K'\bar{L}'} & \Omega_{K\bar{L}}^{\bar{I}'}{}^{J'} &= R_{K\bar{L}}^{\bar{I}'}{}^{J'} \\
\Omega_{\bar{I}}{}^{J'K}{}^{\bar{L}'} &= R_{\bar{I}}{}^{J'K}{}^{\bar{L}'} & \Omega_{\bar{I}}{}^{J'K}{}^{L'} &= (\text{Im } \tau^{-1})^{J'\bar{M}} (\text{Im } \tau^{-1})^{L'\bar{N}} \bar{G}_{\bar{I}\bar{K}\bar{M}\bar{N}}
\end{aligned} \tag{A.2}$$

along with conjugates and terms obtained via the trivial symmetry $\Omega_{ABCD} = -\Omega_{ABDC}$. The tensors $R_{I\bar{J}K\bar{L}}$ and G_{IJKL} are as in (7.8) and (7.9).

B Summary of $SU(1, 1|4)$

$SU(1, 1|4)$ is a simple superalgebra with bosonic part $SO(2, 1) \times U(4)$ and may be represented in terms of $(2|4) \times (2|4)$ supermatrices

$$\left(\begin{array}{c|c} SL(2; \mathbb{R}) & \text{fermions} \\ \hline \text{fermions} & SU(4) \end{array} \right)$$

with a diagonal $U(1)$ factor satisfying $\text{Str} = 0$.

The charges generating $SU(1, 1|4)$ are

$$\begin{aligned}
H &= (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}_{\bar{A}}^{\bar{J}} \psi^{KB} \bar{\psi}_{\bar{B}}^{\bar{L}} \\
&\quad + \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{IJKL} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \\
D &= a^I \Pi_I + \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}} \\
K &= \text{Im} \left(\frac{\partial \mathcal{F}}{\partial a^I} \bar{a}^I \right) \\
\mathcal{R} &= i \left(a^I \Pi_I - \bar{a}^{\bar{I}} \bar{\Pi}_{\bar{I}} \right) + \frac{1}{2} \text{Im } \tau_{I\bar{J}} \psi^{IA} \bar{\psi}_{\bar{A}}^{\bar{J}} \\
R^{A\bar{B}} &= i \text{Im } \tau_{I\bar{J}} \left(\psi^{IA} \bar{\psi}_{\bar{A}}^{\bar{J}\bar{B}} - \frac{1}{4} \delta^{A\bar{B}} \psi^{IC} \bar{\psi}_{\bar{C}}^{\bar{J}} \right) \\
Q^A &= \psi^{IA} \Pi_I + \frac{1}{12} \epsilon_{\bar{B}\bar{C}\bar{D}}^A \bar{\mathcal{F}}_{\bar{I}\bar{J}\bar{K}}^{(3)} \bar{\psi}^{\bar{I}\bar{B}} \bar{\psi}^{\bar{J}\bar{C}} \bar{\psi}^{\bar{K}\bar{D}} & \bar{Q}^{\bar{A}} &= (Q^A)^\dagger \\
S^A &= \text{Im } \tau_{I\bar{J}} \bar{a}^{\bar{J}} \psi^{IA} & \bar{S}^{\bar{A}} &= (S^A)^\dagger.
\end{aligned}$$

The non-vanishing boson-boson commutators are

$$\begin{aligned} [H, K] &= -iD & [D, K] &= -2iK & [D, H] &= 2iH \\ [R^{A\bar{B}}, R^{C\bar{D}}] &= i \left(\delta^{C\bar{B}} R^{A\bar{D}} - \delta^{A\bar{D}} R^{C\bar{B}} \right). \end{aligned}$$

The nonzero fermion charges are

$$\begin{aligned} [D, Q^A] &= iQ^A & [D, S^A] &= -iS^A \\ [\mathcal{R}, Q^A] &= -\frac{1}{2}Q^A & [\mathcal{R}, S^A] &= -\frac{1}{2}S^A \\ [H, S^A] &= -iQ^A & [K, Q^A] &= iS^A \\ [R^{A\bar{B}}, Q^C] &= i \left(\delta^{C\bar{B}} Q^A - \frac{1}{4} \delta^{A\bar{B}} Q^C \right) & [R^{A\bar{B}}, S^C] &= i \left(\delta^{C\bar{B}} S^A - \frac{1}{4} \delta^{A\bar{B}} S^C \right) \end{aligned}$$

as well as those following from conjugation. Finally, the non-vanishing anticommutators are

$$\begin{aligned} \{Q^A, \bar{Q}^{\bar{B}}\} &= \delta^{A\bar{B}} H & \{S^A, \bar{S}^{\bar{B}}\} &= \delta^{A\bar{B}} K \\ \{Q^A, \bar{S}^{\bar{B}}\} &= \frac{1}{2} \delta^{A\bar{B}} (D - i\mathcal{R}) - R^{A\bar{B}}. \end{aligned}$$

C Deformation of $SU(1, 1|4)$ for nonzero fibre momentum

To obtain $SU(1, 1|4)$ invariance it is necessary to truncate to the sector with $P^{I'} = 0$, both for conformal invariance and R-symmetry enhancement. One can still define generators at nonzero fibre momentum, but their algebra no longer closes. Nevertheless it may be instructive to have explicit expressions for their generators and commutation relations.

The generators which are defined differently are the Hamiltonian

$$\begin{aligned} H &= (\text{Im } \tau^{-1})^{I\bar{J}} \Pi_I \bar{\Pi}_{\bar{J}} + \frac{1}{2} R_{I\bar{J}K\bar{L}} \psi^{IA} \bar{\psi}^{\bar{J}} \psi^{KB} \bar{\psi}^{\bar{L}} \\ &+ \frac{1}{12} \text{Re} (\epsilon_{ABCD} G_{I\bar{J}K\bar{L}} \psi^{IA} \psi^{JB} \psi^{KC} \psi^{LD}) \\ &+ \text{Im } \tau_{I'\bar{J}'} P^{I'} \bar{P}^{\bar{J}'} + \frac{1}{2} \text{Re} \left(\mathcal{F}_{I'\bar{J}K}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} P^{I'} \right) \end{aligned}$$

and the supersymmetries

$$\begin{aligned} Q^A &= \psi^{IA} \Pi_I + \frac{1}{12} \epsilon_{\bar{B}\bar{C}\bar{D}}^A \bar{\mathcal{F}}_{\bar{I}\bar{J}\bar{K}}^{(3)} \bar{\psi}^{\bar{I}\bar{B}} \bar{\psi}^{\bar{J}\bar{C}} \bar{\psi}^{\bar{K}\bar{D}} \\ &+ \text{Im } \tau_{I'\bar{J}'} P^{I'} \Omega_{\bar{B}}^A \bar{\psi}^{\bar{J}\bar{B}}. \end{aligned}$$

These are just as in (8.5) and (8.6). In each case these deformations manifestly break $SO(6) \rightarrow SO(5)$ by use of the symplectic form Ω_{AB} , as well as less obviously breaking conformal invariance. The latter breaking is exemplified by the following deformed commutation relations:

$$\begin{aligned}
[D, H] &= 2iH - 2i \operatorname{Im} \tau_{I' \bar{J}'} P^{I'} \bar{P}^{\bar{J}'} \\
&\quad - \frac{i}{2} \operatorname{Re} \left(\mathcal{F}_{I' JK}^{(3)} \Omega_{AB} \psi^{JA} \psi^{KB} P^{I'} \right) \\
[D, Q^A] &= iQ^A - i \operatorname{Im} \tau_{I' \bar{J}'} P^{I'} \Omega_{\bar{B}}^A \bar{\psi}^{\bar{J} \bar{B}} \\
[H, S^A] &= -iQ^A + i \operatorname{Im} \tau_{I' \bar{J}'} P^{I'} \Omega_{\bar{B}}^A \bar{\psi}^{\bar{J} \bar{B}} \\
\{Q^A, S^B\} &= \operatorname{Im} \tau_{I' \bar{J}'} P^{I'} \Omega^{AB} \bar{a}^{\bar{J}}.
\end{aligned}$$

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